INDEFINITE DESCRIPTIONS

Graham PRIEST

1) *Introduction*

Unlike many sentences that are independent of Z. F. (for example, the G. C. H. or various large cardinal axioms) the axiom of choice is obviously true. Consider the following reasoning: Given any non-empty set of non-empty sets x, if we map every member of x, say y, onto a member of y, then this mapping will be a choice function on x. (It is totally irrelevant that there is in general no «effective» way of doing this. Mathematics is not concerned solely with the «effective»).

The problem now arises: can we formalize the above reasoning in a natural way? Clearly, to do this we must have some formal equivalent of the phrase 'a (particular) x with such and such a property'. In other words we need a theory of indefinite descriptions, i.e. the indefinite article. The purpose of this paper is to establish such a theory. I will attack the problem by trying to establish the correct semantics for such terms. Inevitably this will involve us to a certain extent in the old problem of how to handle denotationless singular terms.

2) *Informal Analysis*

Perhaps the first point to consider is whether such terms need special semantics. After all Russell's theory of descriptions may be seen as a claim that definite descriptions need no special semantics since sentences containing definite descriptions cen be paraphrased away in terms of ones not containing them. Is it possible to find such a paraphrase for indefinite descriptions ? The answer to this is «Almost certainly, no». The reason is as follows.

As is well known, if we add Hilbert's ε -operator to the language of set theory and add the axiom

$$(\exists y) y \in x \rightarrow (\varepsilon y)(y \in x) \in x \tag{1}$$

to Z. F., then providing we allow ε -terms to occur in the axiom schemes of Z. F., the Axiom of Choice is provable. (See e.g. Leisenring [1969] p. 105-107). Now Hilbert's ε-symbol behaves in many ways like the indefinite article. Certainly (1) is intuitively correct if we do interpret it this way. Suppose then that for every sentence containing ε-terms, we could give a logically equivalent sentence not containing them; and in such a way that the paraphrase of (1) were a logical truth. Then if the paraphrase is at all reasonable we would be able to paraphrase the proof of the axiom of choice into a corresponding proof in Z. F. But this is impossible. In particular reading '$(\exists x)(Px \wedge \Phi)$' for '$P(\varepsilon x\Phi)$' will not do.

We are forced then to look for semantics of indefinite description. If Φ is a sentence of some language, let us read '$\varsigma x\Phi$' as 'some (particular) x which Φ's,' and let D be the domain of existent objects.

i) *Atomic Sentences*

Consider first the truth value of an atomic sentence '$P(\varsigma x\Phi)$'. If the set X of x's in D which satisfy Φ is non-empty, then '$\varsigma x\Phi$' denotes one such x and '$P(\varsigma x\Phi)$' is true iff P is true of that x. This much is straightforward. If there is no such x, (i.e. X is empty) then we suggest that '$P(\varsigma x\Phi)$' is false. For if there is no man next door, then it can not be true that a man next door is a good chess player or a homicidal maniac. Hence it is false.

Some have of course argued that sentences which contain non-denoting terms are neither true nor false, e.g. Frege, Strawson in [1950] and elsewhere. But apart from the fact that patently, if '$\varsigma x\Phi$' does not denote, then '$\varsigma x\Phi$ does not exist' is true, it has been well argued by Dummett [1973] pp. 413-420 that calling a sentence neither true nor false, rather than just plain false is an empty gesture. To quote from p. 419.

> 'In order then ... to explain the use of sentences to make assertions, all that has to be appealed to is a twofold clas-

> sification of sentences ... into those which could be used to make a correct [i.e. true] utterance, and those that would result in an incorrect [i.e., false] one'.

Now if $\varsigma x\Phi$ does not exist then it can never be correct to assert '$P(\varsigma x\Phi)$'. Hence it is false.

It is interesting that Dummett then goes on (p. 420-429) to argue that we do need to distinguish two different types of falsity (loosely, false because non-existent and false because existent but wrongly applied) in order to provide adequate truth functional semantics for compound sentences. This claim I take to be false. I hope it will be clear by the end of the paper that satisfactory truth value semantics for all sentences can be obtained without such a distinction.

Some people have actually claimed that atomic sentences containing non-denoting terms may actually be true. Hilbert, for example, gave an arbitrary denotation in cases of natural reference failure. This would obviously allow for some atomic sentences to be true. However, this is obviously *ad hoc*.

More interestingly, the following sorts of examples would seem to show that such sentences can be true.

1) The alchemists sought the philosopher's stone.
2) Sherlock Holmes lived in Baker Street.

However, examples like this always depend on non-denoting terms occurring in non-extensional contexts. Now opaque contexts are an interesting but totally separate problem. (They give rise to problems such as failure of Leibniz' law completely independently of whether or not the terms involved denote), and to run it into the problem of non-denotation is to invite confusion.

It may not be clear that the terms in 2) (and others truths of fiction and mythology) are in non-extensional contexts. However, this is not difficult to demonstrate. 2) is not literally true, though 'In the book by Conan Doyle, Sherlock Holmes lived in Baker Street' certainly is. The words 'In the book by Conan Doyle' are absolutely necessary, though often omitted.

Take any work of fiction F, where the main character X, is an actual historical figure who achieves in the book something, Y, he failed to do in real life. Obviously, to suppress the phrase 'In the work of fiction F' from 'In the work of fiction F, X achieved Y' is to invite a contradiction that does not exist.

Further, such a phrase puts any term in its scope into an opaque context. Otherwise, we should have, by Leibniz' law: in the work of fiction F, the character who did not achieve Y, achieved Y. Such a plot would be difficult to write!

We may sum up the discussion of this point by saying if '$\varsigma x\Phi$' does not exist, and 'P' is an ordinary, extensional predicate, '$P(\varsigma x\Phi)$' is false.

ii) *Truth Functions*

Once we have settled the truth value of atomic sentences, we may consider the truth value of sentences containing connectives to be determined by ordinary truth functions. This is certainly the simplest course. It ensures that classical propositional logic holds, and does no great violence to our intuitions.

There is only one point where it may be argued that this gives, intuitively, the wrong value, and this is probably why Dummett feels that ordinary truth tables are inadequate.

According to our analysis, if $\varsigma x\Phi$ does not exist then '$\urcorner P(\varsigma x\Phi)$' is true. Now a number of people have felt that if there is no man next door then 'The man next door is a maniac' and

'The man next door is not a maniac' (6)

are both false. If we ask why this view is held, we are probably told that from 'The man next door is not a maniac' we can logically infer 'There is a man next door'. Thus (6) can not be true if there is no man next door.

We can account for this possibility in two ways. Firstly, if we accept that 'not' in ordinary English can be used as predicate negation as well as sentence negation (as argued, e.g. by Jackendoff (1972) p. 325 ff.), then we can interpret (6) as an atomic sentence with a complex predicate. This would make the inference correct. Predicate negation is easily formalisable and I will not consider this way further.

Alternatively, we can accept that the 'not' in (6) is sentence negation and that

$$\frac{\neg P(\varsigma x\Phi)}{\varsigma x\Phi \text{ exists}}$$

is not a logically valid inference. (After all the more general

$$\frac{\psi(\varsigma x\Phi)}{\varsigma x\Phi \text{ exists}}$$

is certainly not logically valid).

However, this is not to deny that if someone utters 'The man next door is not a maniac' there are grounds for believing that there is a man next door. For people do not usually make such utterance unless there is a man next door. Thus

$$\frac{\text{A utters 'P}(\varsigma x\Phi)\text{'}}{\varsigma x\Phi \text{ exists}}$$

would be a good inductive inference.

Other cases of what Strawson [1952] pp. 175-179 call's presupposition could be handled in a similar way.

Thus it is possible to defend our view that sentences containing connectives may be evaluated with normal truth tables in either of these ways.

iii) *Quantification*

The truth value of quantified sentences is simply determined by the normal quantifier semantics over the domain D of existent entities. The temptation to admit the truth of

'There are things which don't exist, e.g. Pegasus' (7) (e.g. Resher [1959]) should be fought, if 'There are really means what it says. Of course, we may interpret the quantifier in (7) substitutionally and this has additional plausibility in view of Quine's remarks on quantification in [1973]. Indeed it may be argued that Meinong's desire to assert that in some sense, obects that do not exist are existent (Meinong [1907]) can be laid at the door of substitutional quantification. But to follow this would take us too far a field.

Hence '$(\exists y) y = \varsigma x\Phi$' will be true iff '$\varsigma x\Phi$' denotes. It may therefore conveniently be read as '$\varsigma x\Phi$ exists'.

iv) *Identity*

Finally, a further problem arises with identity and indefinite descriptions. When is '$\varsigma x\Phi = \varsigma x\psi$' (1) true ? There are many possible answers. Hilbert's ε operator makes (1) true when Φ and ψ are extensionally equivalent. There seems no intuitive justification for this or for any other view except that the only time we have a guarantee that (1) is true, is when Φ and ψ are the same. This is the condition we will adopt. Hence, which of the objects satisfying Φ is to be denoted by '$\varsigma x\Phi$' will have to depend on Φ itself. How to achieve this will become clear in section 3. Perhaps it is worth pointing out that our semantics can be modified in fairly obvious ways to accomodate Hilbert's view or a number of others.

This concludes the informal analysis of the semantics for indefinite descriptions.

3) We will now give the formal version of the heuristic semantics of the previous section.

In what follows, I will use the following schematic variables:

Φ, ψ, Θ for formulas; x, y, z for variables; t_i for terms; $\bar{t}$ for a sequence of terms.

Let L be a first order language with variables v_i, $i<\omega$, n-place predicates $P_i^n (i<\omega)$ and no constant or function symbols. For simplicity, we will let identity be P_0^2.

Let L^ς be the language L with ς-terms added, i.e. we add the following clauses to the definitions of 'term' and 'formula'.

i) $\varsigma x\Phi$ is a term.
ii) $P_i^n (t_1 \ldots t_n)$ is a formula.

Let *A* be a structure suitable for L. We will extend *A* to a structure suitable for L^{ς}.

Notation

Let the domain of *A* be A, and let V be the set of variables of L. g, h will be maps from V into A and $g(^x/a)$ will denote

$$g - \{(x,g(x))\} \cup \{(x,a)\}.$$

$A \models \Phi\ [g]$ will mean that Φ is true in *A* under the valuation of the free variables given by g.

dp(Φ) (the depth of Φ) is the length of the longest chain of nested ς-terms in Φ. (Hence if $\Phi \in L$, $dp(\Phi) = O$ and if $dp(\Phi) = n$, $dp(P^1_m(\varsigma x\Phi)) = n+1$). The depth of a ς-term «$\varsigma x\Phi$» is dp(Φ).

$\Phi(^x/y)$ means Φ with all free occurrences of «x» replaced by «y». (without any change of bound variables). Now let π be the set of choice functions on A, ($\pi = \{f;\ f\colon P(A) - \{\wedge\} \rightarrow A$ and $f(B)\ \varepsilon\ B\}$).

Let F be the set of formulas of L^{ς}. Then if $\Omega\colon V \times F \times A^v \rightarrow \pi$, (i.e. if $x \in V$, $g \in A^v$, $\Phi \in F$ then $\Omega(x, \Phi, g)$ is a choice function on A), Ω is *suitable* iff

i) If the variables of Φ occur amongst $x, y_1, \ldots, y_n$ and for $1 \leqq i \leqq n$

$$g(y_i) = h(y_i) \text{ then } \Omega(x,\Phi,g) = \Omega(x,\Phi,h);$$

ii) If y is distinct from z and y is free in $\Phi(^x/y)$ then

$$\Omega(z,\Phi(^x/y),g) = \Omega(z,\Phi,g(^x/g(y))).$$

These conditions ensure that $\Omega(x,\Phi,g)$ depends only on the

members of g which are relevant to Φ and not on the particular variable used to denote a parameter.
Let Ω be suitable and let $B = <A, \Omega>$. The domain, B, of B is A. We define $B \models \Phi[g]$ by recursion over $dp(\Phi)$.
if $dp(\Phi) = O$, then $B \models \Phi[g]$ iff $A \models \Phi[g]$
Suppose the definition is made for all Φ of depth $\leqq n$
The definition for depth $n+1$ is itself by recursion over the formation of Φ.
If Φ is atomic, it is of the form $P^n_i(t_1 \dots t_n)$.
For simplicity, let us suppose that only one term t_j is of depth n and that this is $\varsigma x \psi$.

$B \models P^n_i(t_1 \dots \varsigma x\psi \dots t_n)$ [g] iff

$$f^x_\psi(g) \text{ is defined} \quad \text{and} \quad B \models P^n(t_1 \dots z \dots t_n)\ [g(z/f^x_\psi(g))]$$

where z is a variable not occurring in ψ
and $f^x_\psi(g)$ is the partial function defined as follows:

if $\{b\varepsilon B;\ B \models \psi[g(^x/b)]\}$ is non empty

$$f^x_\psi(g) = \Omega(x,\psi,g)\{b\varepsilon B;\ B \models \psi[g(^x/b)]\}$$

$f^x_\psi(g)$ is undefined otherwise.

(Thus $f^x_\psi(g)$ is the interpretation of the term $\varsigma x \psi$ under the evaluation g.) If Φ is $\Theta v \psi$, $\urcorner\psi$ or $(\exists x)\psi$, the clauses of the definition are as usual. A little thought shows that apart from the need for Ω to be suitable, these are the formal equivalent of the heuristic semantics of the previous section. The suitability conditions arise only because we have to deal with open sentences and the parameters are not part of the language. Thus we have the two following lemmas:

Lemma 1

If the free variables of Φ occur amongst y_i $1 \leqq i \leqq n$ and $h(y_i) = g(y_i)$ $1 \leqq i \leqq n$ then

$$B \models \Phi\,[g] \text{ iff } B \models \Phi\,[h].$$

Proof

By induction over the formation of Φ.

Corollary

If Φ is closed then $B \models \Phi[g]$ for all g or for no g; if for all g then write $B \models \Phi$.

Lemma 2

If y is free in $\Phi(^x/y)$,
$B \models \Phi[g(^x/g(y))]$ iff $B \models \Phi(^x/y)[g]$.

Proof By induction over the formation of Φ.

This lemma is necessary to validate axiom vi) of section four.

4. We will now characterize the semantics axiomatically. Let T be a first order theory in the language L. The following theory in the language L^ς is called T^ς. The axioms of T^ς are specified as follows:

i) Any theorem of T is an axiom.

ii) Axioms for the predicate calculus of L^ς

a) $\Phi \rightarrow (\psi \rightarrow \Phi)$
b) $(\Phi \rightarrow (\psi \rightarrow \Theta)) \rightarrow ((\Phi \rightarrow \psi) \rightarrow (\Phi \rightarrow \Theta))$
c) $((\neg\Phi \rightarrow \neg\psi) \rightarrow (\neg\Phi \rightarrow \psi)) \rightarrow \Phi$
d) $(\forall x)\ \Phi \rightarrow \Phi(^x/y)$ (y not bound in $\Phi(^x/y)$)
e) $(\forall x)\ (\Phi \rightarrow \psi) \rightarrow (\Phi \rightarrow (\forall x)\psi)$ (x not in Φ)
f) $(\forall x)x = x$
g) $x = y \rightarrow (\Phi \leftrightarrow \Phi(^x/y))$ (y not bound in $\Phi(^x/y)$)

Axioms for ς-terms

iii) $y = t \rightarrow (P_i^n(\bar{t}) \leftrightarrow P_i^n(y; \bar{t}))$

iv) $P_i^n(\varsigma x\Phi; \bar{t}) \rightarrow (\exists x)\Phi$

v) $(\exists x)\Phi \rightarrow (\exists y)\ y = \varsigma x\Phi$

vi) $y = \varsigma x\Phi \rightarrow \Phi(^x/y)$ (y not free in $\Phi(^x/y)$)

The rules of inference of T^ς are modus ponens and generalization. The following are theorems of T^ς provided that y does not occur in the scope of a ς-term in Θ, and all the variables free in $\varsigma x\Phi$ are free in $\Theta(^y/\varsigma x\Phi)$.

a) $(\exists y)\ y = \varsigma x\Phi \rightarrow (\exists x)\Phi$ (From axiom iv))
b) $y = \varsigma x\Phi \rightarrow (\Theta \leftrightarrow \Theta(^y/\varsigma x\Phi))$
(By induction over the formation of Θ. Axiom iii) is the basis).
c) $(\exists x)\Phi \wedge (\forall y)\Theta \rightarrow \Theta(^y/\varsigma x\Phi)$
(From b) and axiom v))
d) $(\exists x)\Theta \rightarrow \Theta(^x/\varsigma x\Theta)$
(From b) and axioms v), vi)).

The failure of b) — d) for general contexts Θ(y) is not surprising. ς is not an extensional operator and hence there is no reason to suppose that $\varsigma z\Theta$ is the same as $\varsigma z\Theta(^y/\varsigma x\Phi)$, even if $y = \varsigma x\Phi$. Finally, we have the following results.

Theorem 1.

If $A \models T$, Ω is suitable and $B = \langle A,\Omega\rangle$, $B \models T^\varsigma$

Proof

The proof is straightforward using Lemmas 1 and 2.

Corollary 1.

If A is any first order structure and Ω is suitable then axioms ii) — vi) hold in $\langle A,\Omega\rangle$.

Corollary 2.

If T is consistent then so is T^{ς}.

Corollary 3.

T^{ς} is a conservative extension of T.

Proof

Let $\Phi \varepsilon L$ and suppose that $T \nvdash \Phi$
By the completeness of first order logic, we can find a model A of $\{\neg\Phi\} \cup T$. Let Ω be suitable $B = \langle A, \Omega \rangle$. Since $dp(\Phi) = O$, $B \models \neg\Phi$. Hence by Theorem 1, $T^{\varsigma} \nvdash \Phi$.

5. In section four we saw that our axioms were sound. The axioms are also complete in the following sense:

For a given theory T, a sentence Φ of L^{ς} is *valid* if for every $A \models T$ and every suitable Ω

$$B = \langle A, \Omega \rangle \models \Phi.$$

Theorem 2.

For a given T Φ is a theorem of T^{ς} iff Φ is valid.

Proof

As we saw in the previous section, all the provable sentences are valid. We will prove that if Ξ is any set of closed sentences of L^{ς} consistent with respect to T^{ς}, then Ξ has a model. This implies that all valid sentences are provable.

So let Ξ be such a set. Take a countable set A of new constant symbols and extend Ξ to a set of closed sentences Δ of $L^{\varsigma} \cup A$ such that

i) Δ is maximally consistent.
ii) Δ is satured, i.e. if $(\exists x)\Phi \,\varepsilon\, \Delta$ then for some $a \,\varepsilon\, A$, $\Phi(^x/a) \,\varepsilon\, \Delta$.

This is done in the usual way (as in Henkin [1949]).

With every formula of the form $\exists x\Phi$ in Δ we associate an $a \varepsilon A$ as follows:

$$(\exists x)\Phi \rightarrow (\exists y)y = \varsigma x\Phi \, \varepsilon \, \Delta.$$

so for some $a \varepsilon A$

$$a = \varsigma x\Phi \, \varepsilon \, \Delta.$$

We associate with Φ, the a with the lowest index in some fixed well ordering of A. We will denote this by a^x_Φ.

Thus

$$a^x_\Phi = \varsigma x\Phi \, \varepsilon \, \Delta.$$

The following lemma is useful.

Lemma 3.

For every a^x_Φ, $\Phi(^x/a^x_\Phi) \, \varepsilon \, \Delta$.

Proof

$$(\forall y)\ (y = \varsigma x\Phi \rightarrow \Phi(^x/y)) \varepsilon \, \Delta.$$

Since

$$a^x_\Phi = \varsigma x\Phi \, \varepsilon \, \Delta,$$

then

$$\Phi(^x/a^x_\Phi) \, \varepsilon \, \Delta \text{ since } \Delta \text{ is maximally consistent.}$$

Now we can construct the model $B \models \Delta$. Since $\Delta \supseteq \Sigma$, this gives result. Let $\Delta^- = \Delta \cap L$ (i.e. the sentences of Δ with no ς-terms). Then Δ^- is a maximally consistent, saturated set of sentences of L.

If Φ_g is Φ with every variable y free in Φ, replaced by g(y), then we can define in the usual way a model A with domain A for Δ^- such that

$$A \models \Phi[g] \text{ iff } \Phi_g \varepsilon \Delta^-$$

(see Henkin [1949]).

We can now define Ω. We do this by defining a series of functions $\Omega_n (n \varepsilon \omega)$ such that Ω_n is a map from $V \times F_n \times A^v$ into the set of choice functions on A, (where F_n is the set of formulas of depth $\leqq$ n) such that

i) $\Omega_{n+1} \supseteq \Omega_n$.
ii) Ω_n is suitable.
iii) If Ω is any map from $V \times F \times A^v$ into π such that $\Omega \supseteq \Omega_n$, then for every formula Φ of L of depth n, and every g,

$$<A,\Omega> \models \Phi[g] \text{ iff } \Phi_g \varepsilon \Delta.$$

Depth O

Ω_0 is the empty function. If $dp(\Phi) = 0$ then $\Phi \varepsilon \Delta^-$. Hence the result holds by the above.

Depth n + 1

If $dp(\Phi) \leqq n$ $\Omega_n(x,\Phi,g) = \Omega_{n+1}(x,\Phi,g)$.
If $dp(\Phi) = n + 1$, then for $x \varepsilon V$, $g \varepsilon A^y$, if

$$X = \{b \varepsilon A;\ \Phi_{g(x/b)} \varepsilon \Delta\}$$

and X is not empty, then

$$\Omega_{n+1}(x,\Phi,g)X = a^x_\psi,$$

where ψ is Φ with every variable y, free in Φ except x, replaced by g(y). (This is possible since $(\exists x\Phi)g \varepsilon \Delta$)

If $Y \subseteq A$, $Y \neq X$ and Y is non-empty then

$$\Omega_{n+1}(x,\Phi,y)\, Y \varepsilon Y$$

(Say the least member of Y in some fixed well ordering of A.)

a) Ω_{n+1} is a choice function. The only interesting case is when $X = \{b \,\varepsilon\, A;\ \Phi_{g(x/b)} \,\varepsilon\, \Delta\}$ and X is non empty.
Then $(\exists x\Phi)_g \,\varepsilon\, \Delta$, so $(\Phi(^x/a^x_\psi))_g \,\varepsilon\, \Delta$ (by the Lemma 3).

i.e. $\Phi_{g(x/a^x_\psi)}) \,\varepsilon\, \Delta$ i.e. $a^x_\psi \,\varepsilon\, X$.

b) $\Omega_{n+1} \supseteq \Omega_n$ trivially.
c) Ω_{n+1} is suitable. Again, the only interesting case is when

$$X = \{b;\ \Phi_{g(x/b)} \,\varepsilon\, \Delta\} \text{ and } dp(\Phi) = n + 1.$$

i) Now if g and h agree on all the variables free in Φ except x, then

$$\Phi_{g(x/b)} \text{ is the same as } \Phi_{h(x/b)}$$

Hence $X = \{b;\ \Phi_{h(x/b)} \,\varepsilon\, \Delta\}$ and

$$\Omega(x,\Phi,g)X = \Omega(x,\Phi,h)X = a^x_\psi\ .$$

ii) Similarly $\Omega(z,\Phi(^x/y),g) = \Omega(z,\Phi,g(^x/g(y)))$
since $(\Phi(^x/y))_g$ is the same as $\Phi_{g(x/g(y))}$
d) Finally, if $\Omega \supseteq \Phi_{n+1}$, and Θ is of depth $n + 1$

$$<A,\Omega> \models \Theta[g] \text{ iff } \Theta_g \,\varepsilon\, \Delta.$$

The proof is by induction over the formation of Θ. If Θ is $\neg\psi$, $\chi \vee \psi$ or $(\exists x)\psi$, the result follows from the maximal consistency and saturation of Δ. Hence we need only prove it for atomic Θ.
Suppose $B = <A,\Omega> \models \Theta[g]$

and Θ is $\overset{m}{P_i}(t_1 \ldots \varsigma x\Phi \ldots t_m)$ (and we assume for simplicity that only $\varsigma x\Phi$ is of depth n):

Now $B \models P^m_i(\varsigma x\Phi;\bar{t})\ [g] \Rightarrow$

$$f^x_\Phi(g) \text{ is defined and } B \models P^m_i(z;\bar{t})\ [g(^z/f^x_\Phi(g))]$$

but $f^x_\Phi(g) = \Omega(x,\Phi,g)\{b;\ B \models \Phi[g(^x/b)]\}$.

So by induction hypothesis $f^x_\Phi(g) = a^x_\psi$ where ψ is Φ with every variable y free in Φ replaced by $g(y)$, except x.

Hence $B \models P^m(z;\bar{t})\ [g(^z/a^x_\psi)]$

So by induction hypothesis $(P^m_i(a_\psi;\bar{t}))_g \ \varepsilon\ \Delta$.

But $B \models (\exists x)\Phi[g]$. Hence $\exists x\Phi g\ \varepsilon\ \Delta$, and

$$(\varsigma x\Phi)_g = a^x_\psi\ \varepsilon\ \Delta.$$

Hence

$$(P^m_i(\varsigma x\Phi;\bar{t})_g\ \varepsilon\ \Delta,$$

since axiom iii) $\varepsilon\ \Delta$.

Conversely, if not $B \models \Theta[g]$ then if Θ is $P^m_i(\varsigma x\Phi;\bar{t})$ either $f^x_\Phi(g)$ is not defined, or it is defined and $B \models \neg P^m_i(z;t)\ [g(^z/f^x_\Phi(g))]$. If $f^x_\Phi(g)$ is not defined then $B \models \neg(\exists x)\Phi[g]$. Thus $(\neg(\exists x)\Phi)_g\ \varepsilon\ \Delta$ by induction hypothesis.

Since axiom iv) $\varepsilon\ \Delta$, then

$$\neg P^m_i(\varsigma x\Phi;\bar{t})_g\ \varepsilon\ \Delta$$

$$\text{i.e. } P^m_i(\varsigma x\Phi;\bar{t})_g \not\ni \Delta.$$

If $f^x_\Phi(g)$ is defined, then as above $f^x_\Phi(g) = a^x_\psi$

and

$$\neg P^m_i(\varsigma x\Phi;\bar{t})_g \, \varepsilon \, \Delta.$$

So

$$P^m_i(\varsigma x\Phi;\bar{t})_g \not\ni \Delta.$$

Hence the result.

Now if we put $\Omega = \bigcup_{n \varepsilon \omega} \Omega_n$

and

$$B = \langle A,\Omega \rangle$$

then it is easily checked that Ω_1 is suitable and

$$B \models \Phi[g] \text{ iff } \Phi_g \, \varepsilon \, \Delta.$$

So if Φ is closed, Φ is Φ_g and since Ω is suitable,

$$B \models \Phi \text{ iff } \Phi \, \varepsilon \, \Delta.$$

Thus the semantics are complete.

We have the following two corollaries.

Corollary 1.

If Φ is true in all structures of the form $\langle A,\Omega \rangle$ then Φ is provable from axioms ii) — vi).

Corollary 2.

Compactness: if every finite subset of Σ has a model, then so does Σ.

6) *Conclusion*

Thus we conclude the paper. The above system of indefinite descriptions is, I think, an adequate formalisation of our notion of the indefinite article. There are two final comments.

i) We have treated identity as an ordinary predicate. Thus if '$\varsigma x \Phi$' does not denote, '$\varsigma x \Phi = \varsigma x \Phi$' is false. Some people have argued that this is not so. (e.g. Leblanc and Halperin [1959]). However, the arguments are not very convincing and by far the simpler course is the one we have taken. Nevertheless, the above semantics could be adapted to allow for this by simply adding a special clause for identity statements in the definition of '$\models$'.

ii) It may easily be checked that if we add the description axioms to Z.F. (and allow descriptive terms to occur in the Z.F. schemes) then the axiom of choice is indeed provable.

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